1a) Fixed Point Thm: Let $g \in C^1[a,b]$ be such that $g(x) \in [a,b]$ $\forall x \in [a,b]$. Suppose also that a constant $0<k<1$ exists with

$$|g'(x)| \le k < 1 \quad \forall x \in (a,b)$$

then for any $x_0 \in [a,b]$ the sequence defined by $x_{n+1} = g(x_n)$ converges to the unique fixed point in $[a,b]$.

b) i) At a fixed point $x = g(x) \Rightarrow x = g(x) = x + \frac{1}{2}(2-e^x) \Rightarrow 0 = 2 - e^x$
$\Rightarrow e^x = 2 \Rightarrow \underline{x = \ln 2}$

ii) First show $g(x) \in [0,1]$ $\forall x \in [0,1]$. To do this find all extrem values of $g(x)$ on $[0,1]$. which are at $x=0$, $x=1$ & when $g'(x)=0$ but
$g(0) = 0.5$, $g(1) = 1 + \frac{1}{2}(2-e) = 2 - \frac{1}{2}e = 0.6409$ & $g'(x) = 1 - \frac{1}{2}e^x$
so $g'(x) = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^x \Rightarrow e^x = 2 \Rightarrow x = \ln 2$ & $g(\ln 2) = \ln 2$

Since $g(0) \in [0,1]$, $g(1) \in [0,1]$, $g(\ln 2) \in [0,1]$ by extreme value thm $g(x) \in [0,1]$ $\forall x \in [0,1]$.

Next consider $g'(x) = 1 - \frac{1}{2}e^x \Rightarrow g'(0) = \frac{1}{2}$ $g'(1) = 1 - \frac{1}{2}e = -0.3591$
& since $g''(x) = -\frac{1}{2}e^x \neq 0$ $\forall x \in [0,1]$ these are the only extreme values
&
$|g'(x)| \le \frac{1}{2}$ $\forall x \in [0,1]$. Therefor thm is satisfied with $k = \frac{1}{2}$.

iii) since $g'(\ln 2) = 0$ & $g''(\ln 2) \neq 0$ the scheme is quadratically convergent.

iv) $x_{n+1} = x_n + \frac{1}{2}(2 - e^{x_n})$

$x_0 = 0.5$
$\Rightarrow x_1 = 0.67564$
$\Rightarrow x_2 = 0.69299$
$\Rightarrow x_3 = 0.693147169$

v) Relative err $= \left| \dfrac{x_3 - \ln 2}{\ln 2} \right| = \underline{\underline{1.66 \times 10^{-8}}}$

2 a) $f[x_j] = f(x_j)$

b) $w_0(x) = 1$

$w_j(x) = \prod_{i=0}^{j-1} (x - x_i) \qquad j = 1, \ldots, n$

$c_j = f[x_0, x_1, \ldots, x_j] \qquad j = 0, 1, \ldots, n$

c) Let $g(x) = f(x) - p_n(x)$. Since $f(x_i) = p_n(x_i)$ for $i = 0, 1, \ldots, n$ g has $n+1$ zeros, so by generalized Rolle's Thm $g^{(n)}(x)$ has a zero.
So $\exists\, \xi \in [x_0, x_n]$ s.t. $g^{(n)}(\xi) = 0$ so $0 = g^{(n)}(\xi) = f^{(n)}(\xi) - p_n^{(n)}(\xi)$

But $p_n(x) = f[x_0, x_1, \ldots, x_n]x^n +$ lower order terms

$\Rightarrow p_n^{(n)}(x) = n!\, f[x_0, x_1, \ldots, x_n]$

so $0 = f^{(n)}(\xi) - n!\, f[x_0, x_1, \ldots, x_n] \Rightarrow f[x_0, \ldots, x_n] = \dfrac{f^{(n)}(\xi)}{n!}$

d)

	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
$x_0 = 0$	2			
$x_1 = 1$	3	$\frac{3-2}{1-0} = 1$		
$x_2 = 2$	10	$\frac{10-3}{2-1} = 7$	$\frac{7-1}{2-0} = 3$	
$x_3 = 3$	29	$\frac{29-10}{3-2} = 19$	$\frac{19-7}{3-1} = 6$	$\frac{6-3}{3-0} = 1$

i) $p_3(x) = 2 + 1(x-0) + 3(x-0)(x-1) + 1(x)(x-1)(x-2)$

$= 2 + x + 3x(x-1) + x(x-1)(x-2)$ (acceptable)

$= x^3 + 2$

$p_2(x) = 3 + 7(x-1) + 6(x-1)(x-2)$

$= 6x^2 - 11x + 8$

$p_3(2.5) = 2.5^3 + 2 = 17.625$

$p_2(2.5) = 6(2.5)^2 - 11(2.5) + 8 = 18$

3 (a) $$f(x_0+h) = f(x_0) + hf'(x_0) + \tfrac{1}{2}h^2 f''(x_0) + \tfrac{1}{6}h^3 f'''(x_0) + O(h^4)$$

$$\Rightarrow \quad f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \tfrac{1}{2}hf''(x_0) - \tfrac{1}{6}h^2 f'''(x_0) + O(h^3)$$

$$= N_1(h) + c_1 h + c_2 h^2 + O(h^3) \quad \text{as required} \qquad (1)$$

(b) $$f'(x_0) = N_1\left(\tfrac{h}{2}\right) + c_1\left(\tfrac{h}{2}\right) + c_2\left(\tfrac{h}{2}\right)^2 + O(h^3) \qquad (2)$$

So $2(2)-(1) \Rightarrow f'(x_0) = 2N_1\left(\tfrac{h}{2}\right) - N_1(h) - \tfrac{1}{2}c_2 h^2 + O(h^3)$

So letting $N_2(h) = 2N_1\left(\tfrac{h}{2}\right) - N_1(h)$ & $k_2 = -\tfrac{1}{2}c_2$

$$\Rightarrow \quad f'(x_0) = N_2(h) + k_2 h^2 + O(h^3) \quad \text{as required.} \qquad (3)$$

Now $$f'(x_0) = N_2\left(\tfrac{h}{2}\right) + k_2\left(\tfrac{h}{2}\right)^2 + O(h^3) \qquad (4)$$

So $\frac{4(4)-(3)}{3} \Rightarrow f'(x_0) = \frac{4N_2\left(\tfrac{h}{2}\right) - N_2(h)}{3} + O(h^3)$

Letting $N_3(h) = \frac{4N_2\left(\tfrac{h}{2}\right) - N_2(h)}{3}$ gives required result.

c) $N_1(0.4) = 0.97355$
$N_1(0.2) = 0.99335$
$N_1(0.1) = 0.99833$

$N_2(0.4) = 2N_1(0.2) - N_1(0.4) = 1.01315$
$N_2(0.2) = 2N_1(0.1) - N_1(0.2) = 1.00331$

$$N_3(0.4) = \frac{4N_2(0.2) - N_2(0.4)}{3} = 1.00003$$

4a) The degree of accuracy of a quadrature formula I_h is the largest positive integer n s.t.
$I(p) = I_h(p)$ for all polynomials of degree less than or equal to n

b) To determine degree of accuracy consider

$$I(x^n) = \int_0^1 x^n \, dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}$$

whilst $I_h = \frac{3}{2} \cdot \frac{1}{3}\left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right]$ since $x_0 = 0, x_3 = 1 \Rightarrow h = \frac{1}{3}$

$$= \frac{1}{2}\left[\frac{1+2^n}{3^n}\right]$$

$$\Rightarrow I_h(x^0) = \frac{1}{2}\left[\frac{1+1}{1}\right] = 1 = I(x^0)$$

$$I_h(x) = \frac{1}{2}\left[\frac{1+2}{3}\right] = \frac{1}{2} = I(x)$$

$$I_h(x^2) = \frac{1}{2}\left[\frac{1+4}{3^2}\right] = \frac{5}{18} \neq \frac{1}{3} = I(x^2)$$

So degree of accuracy is 1.

c) Since $I(x^2) = I_h(x^2) + kh^3 \frac{d^2}{dx^2}(x^2)$

$$\Rightarrow \frac{1}{3} = \frac{5}{18} + k\left(\frac{1}{3}\right)^3 2 \Rightarrow \frac{1}{18} = \frac{2}{27}k \Rightarrow \underline{\underline{k = \frac{3}{4}}}$$

d) $I_h(f) = \frac{3}{2} \cdot \frac{1}{3}\left[\frac{1}{4/3} + \frac{1}{5/3}\right] = \frac{1}{2}\left[\frac{3}{4} + \frac{3}{5}\right] = \frac{27}{40}$

Now $I(f) = I_h(f) + kh^3 f''(\xi)$

& $h = \frac{1}{3}$, $k = \frac{3}{4}$ $f(x) = \frac{1}{x} \Rightarrow f''(x) = \frac{2}{x^3}$ & $\max_{\xi \in [1,2]} |f''(\xi)| = 2$

So $|I(f) - I_h(f)| \leq \frac{3}{4} \cdot \left(\frac{1}{3}\right)^3 \cdot 2 = \frac{1}{18} = \underline{\underline{0.05556}}$

5a) The Composite Simpson's Rule is derived by applying Simpson's rule to each pair of intervals. Thus

$$I_n(f) = \sum_{j=0}^{\frac{n}{2}-1} \frac{h}{3}\left[f_{2j} + 4f_{2j+1} + f_{2j+2}\right] = \frac{h}{3}\left[f_0 + f_n\right] + \frac{2h}{3}\sum_{j=1}^{\frac{n}{2}-1} f_{2j} + \frac{4h}{3}\sum_{j=0}^{\frac{n}{2}-1} f_{2j+1}$$

b) The error bound follows from summing the errors in each pair of subintervals from Simpson's Rule:

$$I(f) - I_n(f) = \sum_{j=0}^{\frac{n}{2}-1}\left[-\frac{h^5}{90} f^{(4)}(\eta_j)\right] \quad \text{where } \eta_j \in [x_{2j}, x_{2j+2}]$$

$$= -\frac{h^5}{90}\sum_{j=0}^{\frac{n}{2}-1} f^{(4)}(\eta_j)$$

Now assuming $f^{(4)}(x)$ continuous on $[a,b]$: then

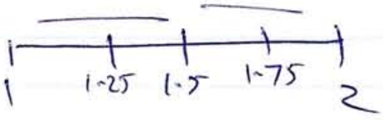
$$m = \min_{x\in[a,b]} f^{(4)}(x) \leq \frac{2}{n}\sum_{j=0}^{\frac{n}{2}-1} f^{(4)}(\eta_j) \leq \max_{x\in[a,b]} f^{(4)}(x) = M$$

but by intermediate value thm $f^{(4)}$ takes all values between m & M & so $\exists\, \xi \in [a,b]$ s.t. $f^{(4)}(\xi) = \frac{2}{n}\sum_{j=0}^{\frac{n}{2}-1} f^{(4)}(\eta_j)$

$$\Rightarrow I(f) - I_n(f) = -\frac{h^5}{90}\cdot\frac{n}{2} f^{(4)}(\xi) \qquad \text{But } nh = b-a \text{ so}$$

$$\underline{\underline{I(f) - I_n(f) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi)}}$$

c) i) $I_{0.25}(f) =$



$$= \frac{0.25}{3}\left[f(1) + f(2)\right] + \frac{2}{3}(0.25)\left[f(1.5)\right] + \frac{4}{3}(0.25)\left[f(1.25) + f(1.75)\right]$$

$$= \frac{0.25}{3}\left[1 + \frac{1}{2}\right] + \frac{2}{12}\left[\frac{2}{3}\right] + \frac{1}{3}\left[\frac{4}{5} + \frac{4}{7}\right]$$

$$= \frac{1}{8} + \frac{1}{9} + \frac{48}{3\cdot 35} = \frac{1}{8} + \frac{1}{9} + \frac{16}{35} = 0.69325.$$

ii) Need $10^{-8} \geq |I(f) - I_n(f)| = \left|\frac{(b-a)h^4 f^{(4)}(\xi)}{180}\right|$ but $b-a = 1$

& $f = \frac{1}{x} \Rightarrow f^{(4)}(x) = \frac{4!}{x^5}$ so $10^{-8} \geq \frac{24}{180} h^4 \left|\frac{1}{x^5}\right| \quad x \in [1,2]$

so $h^4 \leq \frac{180}{24} \times 10^{-8}$ so $h \leq 0.01655$

6 a) Local truncation error is $\tau_{i+1}(h) = y(t_{i+1}) - w_{i+1}$ where

$y(t)$ solves $\begin{cases} y' = f(y) \\ y(t_i) = w_i \end{cases}$ & w_{i+1} is defined by numerical method

For this method

$$w_{i+1} = w_i + \frac{h}{3} f(w_i) + \frac{2h}{3} f\left(w_i + \frac{3h}{4} f(w_i)\right)$$

$$= w_i + \frac{h}{3} f(w_i) + \frac{2h}{3}\left[f(w_i) + \frac{3h}{4} f(w_i) f'(w_i) + \frac{1}{2}\left[\frac{3h}{4} f(w_i)\right]^2 f''(w_i) + O(h^3)\right]$$

$$= w_i + h f(w_i) + \frac{h^2}{2} f(w_i) f'(w_i) + \frac{3h^3}{16} f^2(w_i) f''(w_i) + O(h^4)$$

but exact solution is

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{6} y'''(t_i) + O(h^4)$$

& $y' = f$, $y'' = \frac{d}{dt}[y'] = \frac{d}{dt} f = \frac{dy}{dt}\frac{df}{dy} = ff'$

$y''' = \frac{d}{dt}[ff'] = f(f')^2 + f^2 f''$ so

$$y(t_{i+1}) = y(t_i) + h f(y(t_i)) + \frac{h^2}{2} ff' + \frac{h^3}{6}\left[f(f')^2 + f^2 f''\right] + O(h^4)$$

So letting $y(t_i) = w_i$ $\quad \tau_{i+1} = y(t_{i+1}) - w_{i+1} = O(h^3) \Rightarrow$ Method is 2nd order

6) $f(y) = \lambda y \Rightarrow w_{i+1} = w_i + \frac{h}{3}[\lambda w_i] + \frac{2h}{3}\lambda\left[w_i + \frac{3h}{4}\lambda w_i\right]$

$$= \left(1 + h\lambda + \tfrac{1}{2} h^2\lambda^2\right) w_i \text{ as required}$$

$$= R(h\lambda) w_i$$

For $w_i \to 0$ as $i \to \infty$ need $|R(h\lambda)| < 1$ but

$R(h\lambda) = 1 + h\lambda + \frac{1}{2}h^2\lambda^2 = \frac{1}{2}(h\lambda + 1)^2 + \frac{1}{2} \geq \frac{1}{2}$ so only need to check $R(h\lambda) < 1$

$\cancel{1} + h\lambda + \frac{1}{2}h^2\lambda^2 < \cancel{1}$ $\quad h\lambda\left(1 + \frac{1}{2}h\lambda\right) < 0$ but $h\lambda < 0$ so

$\left(1 + \frac{1}{2}h\lambda\right) > 0$ $\quad h\lambda > -2$ $\quad \boxed{h < \frac{2}{-\lambda}}$ then $w_i \to 0$ as $i \to \infty$

7 a) There exists a unique solution if p, q, r are continuous on $[a,b]$ & $q(t) > 0$ on $[a,b]$.

b) Replace $y'' = ty' + 2y - t \quad 0 \le t \le 1 \quad y(0)=0 \quad y(1)=2$ by

IVP1) $y_1'' = ty_1' + 2y_1 - t$ $\quad$ IVP2) $y_2'' = ty_2' + 2y_2$

$y_1(0) = 0 \quad y_1'(0) = 0$ $\quad$ $y_2(0) = 0 \quad y_2'(0) = 1$

Rewrite as system of equations

IVPS1) let $u = y_1 \quad v = y_1'$ $\quad$ let $p = y_2 \quad q = y_2'$

$\Rightarrow \quad u' = v$

$v' = tv + 2u - t$

$u(0) = 0 \quad v(0) = 0$

$p' = q$

$q' = tq + 2p$

$p(0) = 0 \quad q(0) = 1$

Solve using Euler with $h = 0.5$

IVPS1 $\quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u_1 = u_0 + hv_0 = 0 + (0.5)0 = 0$

$v_1 = v_0 + h[t_0 v_0 + 2u_0 - t_0] = 0 + 0.5[0 + 0 - 0] = 0$

$u_2 = u_1 + hv_1 = 0 + 0 = 0$

$v_2 = v_1 + h[t_1 v_1 + 2u_1 - t_1] = 0 + 0.5[0 + 0 - 0.5] = -0.25$

IVPS2 $\quad p_0 = 0$

$q_0 = 1$

$p_1 = p_0 + hq_0 = 0 + 0.5(1) = 0.5$

$q_1 = q_0 + h[t_0 q_0 + 2p_0] = 1 + 0.5[0 + 0] = 1$

$p_2 = p_1 + hq_1 = 0.5 + 0.5(1) = 1$

$q_2 = q_1 + h[t_1 q_1 + 2p_1] = 1 + 0.5[0.5 + 1] = 1.75$

Now $y(0.5) = y_1(0.5) + \dfrac{y(1) - y_1(1)}{y_2(1)} y_2(0.5) = u_1 + \dfrac{(2 - u_2)}{p_2} p_1 = 0 + \dfrac{(2-0)}{1} 0.5$

$\underline{y(0.5) = 1}$